Long-tem policy-making, Lecture 5

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The Ramsey Problem

An individual or a government tries to allocate resources between investement and consumption to maximize intertemporal welfare

$k(t)$ **capital** at time t, $c(t)$ **consumption** at time t

$f(k)$ instantaneous **production** if capital is k

$f(k(t)) = c(t) + \frac{dk}{dt}$ balance equation

If interest rate is $\rho > 0$, and **utility of consuming** c is $u(c)$, then the problem at time $t = 0$ is:

$$\max \int_0^\infty e^{-\rho t} u(c(t))\, dt,$$

$$\frac{dk}{dt} = f(k(t)) - c(t) \text{ and } k(0) = k_0$$

The optimal consumption path

Under standard assumptions on f and u (concavity and Inada conditions) there is a unique **optimal control** $c(t)$ leading to a **stationary point** k_∞

$$\begin{matrix} \frac{dk}{dt} = f(k(t)) - c(t) \\ k(0) = k_0 \end{matrix} \implies k(t) \longrightarrow k_\infty \quad \forall k_0$$

which satisfies $f(k_\infty) = c_\infty = \sigma(k_\infty)$ and is characterized by

$$f'(k_\infty) = \rho. \text{ independent of } k_0 \text{ and } u(c)$$

This control can be implemented by an **optimal strategy** $c(t) = \sigma(k(t))$. which is given by the HHB equation. In the caseof **logarithmic utility** $u(c) = \ln c$, this is:

$$\begin{aligned} \ln v' - fv' &= \rho v - 1, \quad k \geq 0 \\ \sigma(k) &= \frac{1}{v'(k)}, \end{aligned}$$

Sumaila-Walters discounting

- the population grows at the rate γ
- each generation has a pure rate of time preference ρ
- each generation discounts at the rate $\delta < \rho$ the utility of future generations

For an event which is to happen at time t, we find that the discount factor to apply is:

$$R(t) = e^{-\rho t} + \gamma \int_0^t e^{-\delta s} e^{-\rho(t-s)} ds = (1-\lambda) e^{-\rho t} + \lambda e^{-\delta t}, \text{ with } \lambda = \frac{\gamma}{\rho -$$

Note that this corresponds to a non-constant rate of time preference

- $r(t) = \ln (1-\lambda) e^{-\rho t} + \lambda e^{-\delta t}$
- $r(t) \simeq -\delta$ in the long term
- $r(t) \simeq \lambda\rho + (1-\lambda)\delta$ in the short term

The Ramsey problem revisited

$$\max \int_0^\infty R(t)\, u(c(t))\, dt,$$

$$\frac{dk}{dt} = f(k(t)) - c(t) \text{ and } k(0) = k_0$$

where the discount function $R(t)$ satisfies:

$$R(t) \geq R(0) = 1,\ R(t) \longrightarrow 0 \text{ when } t \longrightarrow \infty$$

There is still a unique optimal control $\bar{c}(t)$, satisfying the Euler equation:

$$\frac{R'(s)}{R(s)} + \frac{u''(c(s))}{u'(c(s))}\frac{dc}{ds} + f'(k(s)) = 0 \quad \text{for} \quad 0 \leq s \qquad \text{(0-}\mathit{optimal}\text{)}$$

Optimal today, suboptimal tomorrow

At some later time $t > 0$, the decision-maker may reconsider the problem:

$$\max \int_t^{\infty} R(t-s)\, u(c(s))\, ds$$
$$\frac{dk}{ds} = f(s, k(s)) - c(s), \; s \geq t, \quad k(t) = k_t$$

There is still an optimal trajectory $s \longrightarrow \left(\tilde{c}(s), \tilde{k}(s)\right)$, starting from $\tilde{k}(t) = \bar{k}(t)$, which satisfies the Euler equation from

$$\frac{R'(s-t)}{R(s-t)} + \frac{u''(c(s))}{u'(c(s))} \frac{dc}{ds} + f'(k(s)) = 0 \;\text{ for }\; t \leq s \qquad (t\textit{-optimal})$$

If $(\bar{c}(t), \bar{k}(t)) = \left(\tilde{c}(s), \tilde{k}(s)\right)$, then this function must satify *both* equations on $s \geq t$. This is clearly not possible unless R is an exponential. The policy $t \longrightarrow (\bar{c}(t), \bar{x}(t))$, which is **optimal** from the time 0 point of view, is **suboptimal** at later times

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- She expects all later ones to apply the strategy σ
- She asks herself if it is in her own interest to apply the same strategy, that is, to consume $\sigma(k(t))$.
- σ is an **equilibrium strategy** if the answer is yes.

Equilibrium strategies: definition

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- On $[t+\varepsilon,\infty]$. the strategy σ takes over, and the new trajectory is $k_\varepsilon(t) = k_0(t) + \varepsilon k_1(t)$

$$\frac{dk_1}{ds} = \left(\frac{\partial f}{\partial k}(k_0(s)) - \frac{\partial \sigma}{\partial k}(k_0(s))\right) k_1(s) \qquad (1)$$

$$k_1(t+\varepsilon) = \sigma(t,k) - c \quad \text{(first variation)} \qquad (2)$$

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- The total gain over $\varepsilon = 0$ is

$$\varepsilon \left[\begin{array}{c} u(c) - u(\sigma(k(t))) \\ + \int_t^\infty R(s-t) \frac{\partial u}{\partial c}(\sigma(k_0(s))) \frac{\partial \sigma}{\partial k}(s, k_0(s))\, k_1(s)\, ds \end{array} \right]$$

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- We shall say that $\sigma : [0,T] \times R^d \rightarrow R^d$ is an **equilibrium strategy** if, for every t and k, the maximum in the brackets is attained for $c = \sigma(k(t))$:

Quasi-exponential discount

From now on, we take the following specification:

$$R(t) = \lambda \exp(-\delta t) + (1-\lambda) \exp(-\rho t)$$

Definition

An equilibrium strategy converges to k_∞ if the corresponding trajectories all converge to k_∞

$$\frac{dk}{dt} = f(k) - \sigma(k), \quad k_\infty = \lim_{t \longrightarrow \infty} k(t) \quad \forall k(0)$$

An existence theorem

Define $\underline{k} \leq \overline{k}$ by:

$$f'\left(\underline{k}\right) = \lambda\delta + (1-\lambda)\,\rho, \quad f'\left(\bar{k}\right) = \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}}$$

Theorem

For every $k \in \left[\underline{k}\ ,\ \bar{k}\right]$, there exists an equilibrium strategy converging to k_∞

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- There are *too many* ! The problem of **rational choice** is not solved yet
- We will show later that converging to the highest capital $\bar{k}$ is the only rational strategy.

Necessary conditions

If such an equilibrium strategy $\sigma(k)$ exists, define the **value functions**:

$$v(k) = \int_0^\infty \left(\lambda e^{-\delta t} + (1-\lambda) e^{-\rho t}\right) u(\sigma(k(t)))\, dt$$
$$w(k) = \int_0^\infty \left(\lambda e^{-\delta t} - (1-\lambda) e^{-\rho t}\right) u(\sigma(k(t)))\, dt$$

They satisfy the following system of ODEs:

$$\left(f - \frac{1}{v'}\right) v' - \ln v' = av + bw$$
$$\left(f - \frac{1}{v'}\right) w' - (2\lambda - 1) \ln v' = bv + cw$$

with $a = (\delta + \rho)/2$ and $b = (\delta - \rho)/2$, and we have:

$$\sigma(k) = 1/v'(k), \quad \sigma(k_\infty) = 1/v'(k_\infty) = f(k_\infty)$$

Sufficient conditions

Theorem

If the system

$$\begin{aligned} \left(f - \frac{1}{v'}\right) v' - \ln v' &= av + bw \\ \left(f - \frac{1}{v'}\right) w' - (2\lambda - 1) \ln v' &= bv + cw \end{aligned}$$

has a C^2 *solution* $v(k), w(k)$ *near* k_∞ *with:*

$$\begin{aligned} v'(k_\infty) &= \frac{1}{f(k_\infty)} \\ av'(k_\infty) + bw'(k_\infty) &= \frac{f'(k_\infty)}{f(k_\infty)} \end{aligned}$$

then $\sigma(k) := 1/v'(k)$ *is an equilibrium strategy converging to* k_∞

Changing unknowns

$$\mu(k) := av(k) + bw(k) - \ln f(k),$$

Take $(v(k), \mu(k))$ instead of $(v(k), w(k))$ as unknowns. The first equation becomes:

$$fv' - 1 - \ln fv' = \mu$$

- if $\mu < 0$ this equation has no solution

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$$fv' - 1 - \ln fv' = \mu$$

- if $\mu < 0$ this equation has no solution
- if $\mu = 0$ the only solution is $fv' = 1$
- if $\mu > 0$ it has two solutions $fv' = 1 + x_i(\mu), i = 1, 2$, with $x(0) = 0, \quad x(\mu) \sim \pm\sqrt{\mu}$

Where the wild things are

$$\frac{dv}{dk} = \frac{1+x(\mu)}{f(k)} \quad \mu(k) \geq 0$$

$$\frac{d\mu}{dk} = \frac{1}{f(k)} \frac{1+x(\mu)}{x(\mu)} D(k,\mu,v) + a\frac{1+x(\mu)}{f(k)} - \frac{f'(k)}{f(k)}$$

$$D = a\mu + \left(b^2 - a^2\right) v + a \ln f(k) + (2\lambda - 1)\, b \ln \frac{1+x(\mu)}{f(k)}$$

- Initial condition:

$$\mu(k_\infty) = 0, \quad v(k_\infty) = \frac{a - (2\lambda - 1)\, b}{a^2 - b^2} \ln f(k_\infty), \quad x(0) = 0$$

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- Bearing in mind that x is not smooth $x(\mu) \sim \pm\sqrt{\mu}$

Changing variables

Take x as the independent variable, so $\mu = x - \ln(1+x)$

$$\begin{aligned}\frac{dk}{dx} &= f(k)\frac{x^2}{1+x}\frac{1}{D(x,k,\nu)},\\ \frac{d\nu}{dx} &= x^2\frac{1}{D(x,k,\nu)}\end{aligned}$$

- $D(x,k,v) = \left(2ax + \left(b^2 - a^2\right)v + \left[(2\lambda - 1)b - a\right]\ln\frac{1+x}{f}\right)(1+x) - xf$

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- $D(x,k,v) =$
  $\left(2ax + \left(b^2 - a^2\right)v + \left[(2\lambda - 1)b - a\right]\ln\frac{1+x}{f}\right)(1+x) - xf$
- Initial condition:

$$x(0) = 0, \quad D(0, k(0), \nu(0)) = 0$$

Blowup

We introduce a new variable s, and the system becomes

$$\begin{aligned}\frac{dx}{ds} &= D(x,k,v)\\ \frac{dk}{ds} &= f(k)\frac{x^2}{1+x}\\ \frac{dv}{ds} &= x^2\end{aligned}$$

The linearized system near $(0, k(0), \nu(0))$ then is:

$$\frac{d}{dt}\begin{pmatrix} x\\ k\\ v\end{pmatrix} = \begin{pmatrix} a+(2\lambda-1)b-f' & (a-(2\lambda-1)b)\frac{f'}{f} & b^2-a^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} x\\ k\\ v\end{pmatrix}$$

Concluding the proof

If $a + (2\lambda - 1)\, b \neq f'(k_\infty)$, we can use the central manifold theorem. If $a + (2\lambda - 1)\, b = f'(k_\infty)$, we must to a further blowup. In all cases, we find a smooth solution(possibly several) with:

$$
\begin{aligned}
v'(k_\infty) &= \frac{1}{f(k_\infty)}, \\
w'(k_\infty) &= \frac{1}{f(k_\infty)} \frac{2f'(k_\infty) - (\delta + \rho)}{\delta - \rho} \\
v''(k_\infty) &= \frac{(\delta - f'(k_\infty))(f'(k_\infty) - \delta)}{f'(k_\infty) - \lambda\delta - (1-\lambda)\rho} \frac{1}{f^2(k_\infty)}
\end{aligned}
$$

More about the degenerate case

Computing $D(x,k,v)$ when $f'(k_\infty) = a + (2\lambda - 1)b$, we get:

$$\begin{aligned}
\frac{dx}{ds} &= \alpha(x)x^2 + \beta(k)(k-k_\infty) + \gamma(k)(k-k_\infty)x + \left(b^2 - a^2\right)(1+x)(
\\
\frac{dk}{ds} &= x^2 \frac{f(k)}{1+x} \\
\frac{dv}{ds} &= x^2
\end{aligned}$$

where α, β, γ are smooth functions of one variable such that:

$$2\alpha(0) = 3a + (2\lambda - 1)b,\ \beta(0) = ((2\lambda - 1)b - a)\frac{f'(k_\infty)}{f(k_\infty)},\ \gamma(0) = \beta(0) -$$

We then perform the change of variables $K(s) := (k(s) - k_\infty)x(s)^{-2}$ and $V(s) := (v(s) - v_\infty)x(s)^{-2}$ One finds $\frac{dK}{ds}(0) = f(\bar{k}) \neq 0$, one can take K instead of s as the independent variable, and we get a regular system of ODEs for $x(K)$ and $V(K)$

Estimates

Linearizing the equation of motion $\frac{dk}{dt} = f(k) - \frac{1}{v'}$:

$$\frac{dx}{dt} = \left(f'(k_\infty) + \frac{v''(k_\infty)}{v'(k_\infty)^2} \right) x$$

Convergence to k_∞ requires that $f'(k_\infty) + \frac{v''(k_\infty)}{v'(k_\infty)^2} \leq 0$, hence:

$$\lambda\delta + (1-\lambda)\rho \leq f'(k_\infty) \leq \frac{1}{\frac{\lambda}{\delta} + \frac{1-\lambda}{\rho}}$$

Stability analysis

Take a point k_∞. At a neighbouring point $k_\varepsilon = k_\infty + \varepsilon$ compare two equilibrium strategies

1. σ_0 converging to k_∞
2. σ_ε converging to k_ε

$$\begin{aligned} v_0 &= \left(\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}\right) \ln f\left(k_\infty\right) + v_0'\left(k_\infty\right)\varepsilon \\ v_\varepsilon &= \left(\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}\right) \ln f\left(k_\varepsilon\right) \\ v_\varepsilon - v_0 &= \left[\left(\frac{\lambda}{\delta} + \frac{(1-\lambda)}{\rho}\right) f'\left(k_\infty\right) - 1\right] \frac{\varepsilon}{f\left(k_\infty\right)} \end{aligned}$$

The bracket is negative, so v_ε is a superior strategy if $\varepsilon > 0$, which is always possible unless $k_\infty = \bar{k}$ (so that k_ε is not an allowable point for convergence)

Rational behaviour again

$$\int_0^\infty [\lambda \exp(-\delta t) + (1-\lambda)\exp(-\rho t)]\, u(c(t))\, dt,$$
$$\frac{dk}{dt} = f(k(t)) - c(t) \text{ and } k(0) = k_0$$

This has infinitely many equilibrium strategies, each of them converging to some $k \in [\underline{k},\ \bar{k}]$

- If $k < \bar{k}$, future generations will eventually find $\bar{k}$ more advantageous

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- The whole strategy then unravels from the end: it is not credible

Rational behaviour again

$$\int_0^\infty \left[\lambda \exp\left(-\delta t\right) + \left(1-\lambda\right) \exp\left(-\rho t\right)\right] u\left(c\left(t\right)\right) dt,$$
$$\frac{dk}{dt} = f\left(k\left(t\right)\right) - c\left(t\right) \text{ and } k\left(0\right) = k_0$$

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- If $k < \bar{k}$, future generations will eventually find $\bar{k}$ more advantageous
- The whole strategy then unravels from the end: it is not credible
- **Rational choice** is $k = \bar{k}$